Conjectures on the number of Langford sequences

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Abstract

We propose an asymptotic formula for the number of Langford sequences (and the Skolem sequences): $L(s,n) \sim n!\,\mathrm{e}^{-\ell(s)n}$, *i.e.* the exponential coefficient $\ell(s,n)$ converges to a constant when $n \to \infty$. For $\ell(2,n)$, we find a numerical convergence to Taniguchi's constant, which probably indicates that further connections may exist between Langford sequences and other mathematical theories.

1 Introduction

The original Langford pairing problem can be traced back to Langford's 1958 submission to *Mathematical Gazette* [1, 2]. It has been generalized to sequences with any number n of colors and any number s of blocks having the same color [3] (at that time, we didn't know that the multiple permutation we considered is called Langford's problem):

> Let $[n]$ represent $\{1, 2, \ldots, n\}$ and $\mathfrak{S}_n$ denote the set of all its permutations. Given a multiset $S = [n]^s$ with the multiplicity s, find the set of all its permutations denoted by $\mathfrak{S}_n^s$, in which any two identical elements are separated by as many terms as their own value. For instance, 3 1 2 1 3 2 is the simplest case. To put it differently, we use $\varpi(i)$ to represent the i-th element of a sequence. Then $\mathfrak{S}_n^s$ is isomorphic to $\{\mathcal{S} : \mathcal{S} \in \mathfrak{S}_{sn}\}$, for whose elements the following conditions hold
>
> $$\begin{cases} \varpi(i) + n = \varpi\big(i + \varpi(i) + 1\big), & \text{if } i + \varpi(i) + 1 \leqslant sn \\ \varpi(i) - n = \varpi\big(n + i - \varpi(i) - 1\big), & \text{otherwise} \end{cases}$$
>
> For the convenience of discussion, we would like to exert another restriction: if two solutions, such as 4 1 3 1 2 4 3 2 and 2 3 4 2 1 3 1 4, can be mutually generated by an inversion of each other, only the one with smaller index of the first n or the same index but greater initial element is counted. The notation $L(s,n)$ will be consistently used to represent the result of such an enumeration.

The number of solutions to Langford's problem up to a symmetry, denoted as $L(2,n)$, can be computed using the Miller backtrack method, the Godfrey algebraic method and the Larsen inclusion-exclusion method [4–7].

2 Conditions for Solvability

In this section, we will give a summary of known results on conditions for solvability. The following statements are quoted from the Ref. [3]:

Theorem 1. *If s is a prime, then the necessary condition for $L(s,n) \neq 0$ is that $n \equiv (-1, 0, \ldots, s-2) \mod s^2$.*

Theorem 2. *The necessary condition for $L(4,n) \neq 0$ is $n \equiv (-1, 0) \mod 8$.*

Conjecture 3. *The condition in Theorem 1 is also sufficient for large enough n's.*

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The only known solution for $L(4,n)$ is $L(4,24)=3$ computed by Richard Noble in 2004 according to the record in Ref. [2]:

4 22 9 24 14 4 6 18 20 7 4 23 9 6 15 4 21 7 12 14 6 19 9 13 22 7 18 6 24 20 15 12 9 7 14 23 5 13 21 16 17 19 5 11 12 18 15 22 5 14 20 13 10 24 5 11 16 12 17 23 21 19 15 10 18 13 2 11 8 2 22 20 2 16 10 2 17 8 24 11 3 19 21 23 3 10 8 1 3 1 16 1 3 1 17 8

10 12 13 19 16 14 1 18 1 24 1 10 1 17 12 20 13 23 6 8 14 16 10 19 22 6 18 12 8 21 13 17 6 10 24 14 20 8 16 6 12 23 15 19 13 18 8 22 5 17 14 21 9 7 5 16 11 20 15 24 5 7 9 19 18 23 5 17 11 7 22 4 9 21 15 3 4 7 20 3 11 4 9 3 24 2 4 3 2 23 15 2 11 22 2 21

14 5 15 19 17 11 4 5 23 21 13 4 7 5 24 14 4 11 15 5 7 4 17 19 13 20 22 9 7 11 14 21 23 10 15 18 7 9 13 24 17 11 16 19 10 14 20 9 12 22 15 8 13 21 18 10 23 9 17 16 8 12 2 19 24 2 10 20 2 8 6 2 22 18 12 21 16 6 8 1 23 1 3 1 6 1 3 12 20 24 3 6 18 16 3 22

For any $s \geqslant 5$, we still have no known solutions for $L(s,n) \neq 0$.

3 Asymptotic Formula

To our knowledge, there is no known formula for $L(s,n)$. In Ref. [3], we conjectured that $L(2,n) \sim n!/2^n$ and $L(3,n) \sim n!/2^{2n+1}$. The following are extended from the previous conjectures:

Conjecture 4. *The number of Langford sequences $L(s,n)$ has the following asymptotic formula*

$$L(s,n) \sim n!\,\mathrm{e}^{-\ell(s)n}, \tag{1}$$

where $\ell(s)$ is an exponential coefficient only dependent on s, i.e.

$$\ell(s,n) = \frac{1}{n}\log\frac{n!}{L(s,n)} \tag{2}$$

converges to a constant when $n \to \infty$. As a variant of Langford sequences, Skolem sequences also have the same asymptotic formula

$$V(s,n) \sim n!\,\mathrm{e}^{-v(s)n}. \tag{3}$$

Conjecture 5. *The exponential coefficient $\ell(2,n)$ for the number of Langford sequences $L(2,n)$ converges to Taniguchi's constant [8]*

$$\lim_{n\to\infty}\ell(2,n) = \prod_{p\in\mathbb{P}}\left(1-\frac{3}{p^3}+\frac{2}{p^4}+\frac{1}{p^5}-\frac{1}{p^6}\right) \tag{4}$$

$$= 0.678\,234\,491\cdots, \tag{5}$$

where the product is over the primes $\mathbb{P}$. More precisely, the specific value can be approximated as

$$\ell(2,n) \simeq \prod_{i=1}^{N}\left(1-\frac{3}{p_i^3}+\frac{2}{p_i^4}+\frac{1}{p_i^5}-\frac{1}{p_i^6}\right), \tag{6}$$

where $p_i \in \mathbb{P}$ is the ith prime number and N is monotonely increasing with n.

These conjectures are numerically supported by the calculations in Table 1–4. We also have computed the exponential coefficents in Table 5 for the planar Langford sequences proposed by Donald E. Knuth [9], but we are not very confident about whether it is convergent or not.

4 Conclusion

In this article, we have summaried the conditions for solvability on Langford sequences and proposed an asymptotic formula for the number of Langford sequences (and the Skolem sequences). Conjecture 5 probably indicates that further connections may exist between Langford sequences and other mathematical theories. We need deeper insights to Langford's problem [2].

n	exact		approximate		error
	$L(2,n)$	$\ell(2,n)$	$\ell(2,n)$	$L(2,n)$	
3	1	0.597 253	0.765 625	1	$\sim 0\%$
4	1	0.794 513		1	$\sim 0\%$
7	26	0.752 438		24	-7.7%
8	150	0.699 246	0.701 560	147	-2.0%
11	$17,792$	0.701 437		1.777×10^4	-0.1%
12	$108,144$	0.699 666		1.057×10^5	-2.2%
15	$39,809,640$	0.693 310	0.687 148	4.367×10^7	$+9.7\%$
16	$326,721,800$	0.691 703		3.514×10^8	$+7.6\%$
19	$256,814,891,280$	0.687 803		2.600×10^{11}	$+1.3\%$
20	$2,636,337,861,200$	0.686 760		2.616×10^{12}	-0.8%
23	$3,799,455,942,515,488$	0.684 045	0.681 745	4.006×10^{15}	$+5.4\%$
24	$46,845,158,056,515,936$	0.683 296		4.862×10^{16}	$+3.8\%$
27	$111,683,611,098,764,903,232$	0.681 309		1.104×10^{20}	-1.2%
28	$1,607,383,260,609,382,393,152$	0.680 745	0.680 305	1.627×10^{21}	$+1.2\%$
31				5.701×10^{24}	
32				9.240×10^{25}	
35			0.679 426	4.861×10^{29}	
36				8.871×10^{30}	

Table 1: Number of Langford sequences $L(2,n)$, OEIS A014552. In Ref. [5], the approximate values $L(2,31) \simeq 5.381 \times 10^{24}$ and $L(2,32) \simeq 8.812 \times 10^{25}$ are obtained using a parallel tempering algorithm, which correspond to the exponential coefficients $\ell(2,31) \simeq 0.682\,171$ and $\ell(2,32) \simeq 0.681\,788$.

References

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n	$V(2,n)$	$v(2,n)$
4	3	0.519 860
5	5	0.635 611
8	252	0.634 397
9	1328	0.623 378
12	$227,968$	0.637 521
13	$1,520,280$	0.639 828
16	$700,078,384$	0.644 072
17	$6,124,491,248$	0.645 265
20	$5,717,789,399,488$	0.648 051
21	$61,782,464,083,584$	0.648 833
24	$102,388,058,845,620,672$	0.650 716
25	$1,317,281,759,888,482,688$	0.651 260
28	$3,532,373,626,038,214,732,032$	0.652 625
29	$52,717,585,747,603,598,276,736$	0.653 028

Table 2: Number of Skolem sequences $V(2,n)$, OEIS A059106. In Ref. [5], the approximate values $V(2,32) \simeq 2.213 \times 10^{26}$ and $V(2,33) \simeq 3.614 \times 10^{27}$ are obtained using a parallel tempering algorithm, which correspond to the exponential coefficients $v(2,32) \simeq 0.653\,012$ and $v(2,33) \simeq 0.654\,541$.

n	$L(3,n)$	$\ell(3,n)$
9	3	1.300 357
10	5	1.349 497
17	$13,440$	1.411 711
18	$54,947$	1.415 629
19	$249,280$	1.416 503

Table 3: Number of Langford sequences $L(3,n)$ [2,3].

n	$V(3,n)$	$v(3,n)$
9	9	1.178 289
10	20	1.210 868
11	33	1.273 255
18	$200,343$	1.343 759
19	$869,006$	1.350 778
20	$4,247,790$	1.353 685

Table 4: Number of Skolem sequences $V(3,n)$ [2].

n	$P(2,n)$	$p(2,n)$
3	1	0.597 253
8	4	1.152 289
11	16	1.339 065
12	40	1.358 195
15	194	1.508 761
16	274	1.566 171
19	2384	1.661 229
20	4719	1.693 813
23	31,856	1.792 943
24	62,124	1.822 827
27	426,502	1.910 895
28	817,717	1.938 410
31	5,724,640	2.017 159
32	10,838,471	2.042 480
35	75,178,742	2.114 308
36	142,349,245	2.137 386
39	977,964,587	2.203 353
40	1,850,941,916	2.224 542

Table 5: Number of planar Langford sequences $P(2,n)$, OEIS A125762. The exponential coefficent is defined as $p(2,n) = \frac{1}{n} \log \frac{n!}{P(2,n)}$.